SEMINAR ON TELESCOPE CONJECTURE

1. Introduction

Chromatic homotopy theory is designed as a miracle relationship between stable homotopy theory and 1-dimentional formal groups. Systematic theories were raised by mathematicians in the last century, among which the most important one is Ravenel's conjectures. Three of them were proven, while the fourth one, telescope conjecture, remained unresolved until last year.

To illustrate our idea more precisely, let's consider the Morava K-theory of height $n+1$ denoted by $K(n+1)$. According to periodicity theorem, for a finite p-local spectrum U of type n+1, there exists a self map $v : \Sigma^k U \to U$ such that it is an isomorphism after tensoring $K(n+1)$. Define $T(n+1) = U[v^{-1}]$, then we could obtain another monochromatic localization $L_{T(n+1)}$.

Question 1.1. What's the relationship between these two natural but competing localizations?

By virtue of nilpotence theorem, we know a fundamental fact is that

$$\mathrm{Sp}_{K(n+1)} \subset \mathrm{Sp}_{T(n+1)}.$$

Stimulated by some calculational observation, Ravenel put forward his famous conjecture, telescope conjecture, which claims that this inclusion should be equality. However, it was still Ravenel himself who began to believe this conjecture is false for general cases in the 1990s. Confronted with this problem, mathematicians defined and studied a series of wonderful properties to analyze the difference between the two categories above. Although their attempts failed, we could thus understand stable homotopy more deeply.

Finally, based on huge quantities of preliminaries, the authors in [BHLS23] construct a K-theoretic counterexample to telescope conjecture. This seminar is intended to study the relevant theories and give a complete disproof.

Let me conclude their ideas inhibited in the article at the end of this section.

The ambidexterity phenomenon introduces higher roots of unity and thus cyclotomic extensions of $T(n+1)$-localized sphere spectrum. The largest cyclotomic extension $\mathbb{S}_{T(n+1)}[\omega_{p^\infty}^{(n+1)}]$ would define an intermediate localization called cyclotomic completion

$$\mathrm{Sp}_{K(n+1)} \subset (\mathrm{Sp}_{T(n+1)})^\wedge_{\mathrm{cyc}} \subset \mathrm{Sp}_{T(n+1)}.$$

Cyclotomic completion is much easier to handle. In particular, we will show that for a common example in chromatic homotopy theory $\mathrm{BP}\langle n\rangle$ with a delicate Adams operation,

$$L_{T(n+1)}K(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})^\wedge_{\mathrm{cyc}} \cong L_{T(n+1)}K(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}} \tag{1.2}$$

by cyclotomic redshift and descent theory. Since homotopy fixed points don't always commute with functors like this, maybe we could obtain a counterexample, i.e.,

Theorem 1.3. *Let p be any prime and $n+1 \geq 2$. Then, for all $k \geq 0$, the spectrum*

$$L_{T(n+1)}K(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}})$$

is $T(n+1)$-local but not cyclotomically complete. In other words, the coassembly map

$$L_{T(n+1)}K(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \to L_{T(n+1)}K(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}}$$

isn't an equivalence. Hence, telescope conjecture is false, and cyclotomic complete spectra are strictly contained in $T(n+1)$-local ones.

In order to solve the difficulty brought by algebraic K-theory, we should relate it to topological cyclic homology TC. Thanks to variants of the DGM theorem, it suffices to show that the coassembly

$$L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \to L_{T(n+1)}\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}} \tag{1.4}$$

isn't an equivalence.

The next reduction comes from the asymptotic constancy of $\mathrm{BP}\langle \mathrm{n}\rangle$, helping us simplify the complexity.

Theorem 1.5. *For all k great enough, we have the following commutative diagram*

$$\begin{tikzcd}
T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\cong"] & T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{hp^k\mathbb{Z}} \arrow[d, "\cong"] \\
T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle^{B\mathbb{Z}}) \arrow[r] & T(n+1)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle)^{B\mathbb{Z}},
\end{tikzcd}$$

where horizontal maps are coassembly maps and $B\mathbb{Z}$ means the homotopy fixed point of trivial action.

However, the following two classical theorems suggest the fiber of the lower horizontal map couldn't vanish.

Theorem 1.6. *For any p-complete $\mathbb{E}_1$-ring R, the p-completion of $\mathrm{TC}(R)$ is in the thick subcategory generated by the p-completion of the fiber of $\mathrm{TC}(R^{B\mathbb{Z}}) \to \mathrm{TC}(R)^{B\mathbb{Z}}$.*

Theorem 1.7. *$T(n+1) \otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle)$ doesn't vanish.*

2. Schedule

This seminar is divided into 12 relevant talks. The mission of each talk is as follows. I hope every participant could have a better grasp of chromatic homotopy through this seminar.

Update: Lecture notes can be seen in our Wechat group and some of the videos are uploaded here.

Talk 1 - Review of Ambidexterity and Chromatic Cyclotomic Extensions. Briefly repeat the main players in ambidexterity. Help the audience recall the definition of higher cyclotomic extensions and the corresponding class field theory. Main references are [Har20] and [CSY21].

Talk 2 - Chromatically Localized Algebraic K-Theory. Recall the common definition of (equivariant) algebraic K-theory spectrum. Explain the dévissage process to prove the purity theorem and descent theorem for p-groups. Verify the corollaries of vanishing theorems. If time permits, please show the reason why we could reduce K-functors to TC. Main references are [CMNN23] and [LMMT20].

Talk 3 - Cyclotomic Redshift. State and prove the theorems about descent for higher p-groups, cyclotomic redshift, and cyclotomic hyperdescent. The cyclotomic completion of telescopically localized K-theory should be contained at least. Main reference is [BMCSY23].

Talk 4 - Topological Cyclic Homology. Paraphrase Nikolaus and Scholze's explanation of THH and TC. Main reference is [NS18].

Talk 5 - Cyclotomically Complete Spectra. Introduce the notion of affineness, Fourier transform, categorification, and $\mathfrak{R}$-orientations. Use them to obtain the explicit form of cyclotomic completion. Main reference is [BCSY22].

Talk 6 - Spherical Witt Vectors, Chromatic Nullstellensatz, and Redshift Conjecture. Explain the freeness and cofreeness of Lubin-Tate theories. Construct the zeroes by studying the lifting problems with respect to certain morphisms. Give the final proof of redshift conjecture based on calculational outcomes. Main reference is [BSY22].

Talk 7 - Preliminaries. Prove Theorem 1.7. Introduce hcrings, almost compact objects, and locally unipotent $\mathbb{Z}$-actions. Main reference is [HW22] and the appendix of [BHLS23].

Talk 8 - Boundedness of Cyclotomic Spectra. Introduce several boundedness and compactness indices, including weak canonical vanishing condition, strong canonical vanishing condition, exponent of nilpotence, Segal condition, and Bökstedt class. Main reference is section 2 of [BHLS23].

Talk 9 - Interpretation of THH of Cochain over the Circle. Use spherical Witt vectors and free loop spaces to control $\mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ and then prove Theorem 1.6. Main reference is section 3 of [BHLS23].

Talk 10 - Telescopic Asymptotic Constancy. Show the cyclotomic asymptotic constancy based on strong boundedness condition. The lecturer should explicitly explain the machinary of Dehn twist trivialization and the reasons for upgrading it. Main reference is section 4 of [BHLS23].

Talk 11 - Adams Operations on BP$\langle n\rangle$. Repeat the process of constructing $\mathbb{E}_4$ complex orientation of Lubin-Tate theories. Pay more attention to how we obtain the Adams operation on $\mathrm{BP}\langle n\rangle$ by attaching cells and overcoming the obstruction. Main reference is section 5 of [BHLS23].

Talk 12 - Summary and the Final Disproof. Combine all the knowledge from this seminar to show (1.2), reduction to (1.4), and Theorem 1.5. Main reference is section 6 of [BHLS23].

-	Date	Topic	Speaker
Talk 1	April 15	Ambidexterity and chromatic cyclotomic extension	Zhenpeng Li
Talk 2	April 22	Chromatically localized algebraic K-theory	Zhenpeng Li
Talk 3	April 29	Cyclotomic redshift	Zhenpeng Li
Talk 4	May 6	Topological cyclic homology	Daming Zhou
Talk 5	May 13	Chromatic nullstellensatz	Zhenpeng Li
Talk 6	May 20	Cyclotomically complete spectra	Zhenpeng Li
Talk 7	May 27	Preliminaries	Zhenpeng Li
Talk 8	June 3	Boundedness of cyclotomic spectra	Zhenpeng Li
Talk 9	June 10	THH of cochain over the circle	Zhenpeng Li
Talk 10	June 17	Telescopic asymptotic constancy	Zhenpeng Li
Talk 11	June 24	Adams operations	Zhenpeng Li
Talk 12	July 1	Final disproof	Zhenpeng Li

References

[BCSY22] Tobias Barthel, Shachar Carmeli, Tomer M Schlank, and Lior Yanovski. The chromatic fourier transform. *arXiv preprint arXiv:2210.12822*, 2022.

[BHLS23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M Schlank. k-theoretic counterexamples to ravenel's telescope conjecture. *arXiv preprint arXiv:2310.17459*, 2023.

[BMCSY23] Shay Ben-Moshe, Shachar Carmeli, Tomer M Schlank, and Lior Yanovski. Descent and cyclotomic redshift for chromatically localized algebraic k-theory. *arXiv preprint arXiv:2309.07123*, 2023.

[BSY22] Robert Burklund, Tomer M Schlank, and Allen Yuan. The chromatic nullstellensatz. *arXiv preprint arXiv:2207.09929*, 2022.

[CMNN23] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. Descent and vanishing in chromatic algebraic k-theory via group actions. *arXiv preprint arXiv:2011.08233*, 2023.

[CSY21] Shachar Carmeli, Tomer M Schlank, and Lior Yanovski. Chromatic cyclotomic extensions. *arXiv preprint arXiv:2103.02471*, 2021.

[Har20] Yonatan Harpaz. Ambidexterity and the universality of finite spans. *Proceedings of the London Mathematical Society*, 121(5):1121–1170, 2020.

[HW22] Jeremy Hahn and Dylan Wilson. Redshift and multiplication for truncated brown–peterson spectra. *Annals of Mathematics*, 196(3):1277–1351, 2022.

[LMMT20] Markus Land, Akhil Mathew, Lennart Meier, and Georg Tamme. Purity in chromatically localized algebraic -theory. *Journal of the American Mathematical Society*, 2020.

[NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math*, 221(2):203–409, 2018.